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Some results in asymptotic fixed point theory

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Dedicated to Felix Browder on the occasion of his 80th birthday

Abstract. In 1944, Levinson ([22]) introduced the concept of dissipativeness for a map T in a finite-dimensional space which leads to the existence of a fixed point of some iterate T^n for n large, rather than a fixed point of T. Browder ([3]) gave an asymptotic fixed point theorem which proved that T itself had a fixed point. Although Browder's result was a big step, it was not suitable for hyperbolic PDEs and neutral functional differential equations because, in those cases, the map T is not compact. For α-contraction maps the result was extended by Nussbaum ([25]) and Hale and Lopes ([13]) using different methods. In this paper, we review these ideas and some more recent applications.

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In 1944, Levinson ([22]) studied some properties of the dynamics of the van der Pol equation under the influence of a periodic forcing function of period ω. Actually, he considered more general ordinary differential equations (ODE) in $\mathbb{R}^N$ with the coefficients of the vector field being periodic in time of minimal period ω. One of the first problems of concern is the existence of an ω-periodic solution of the ODE. To avoid infinity from having too much influence on the problem, he introduced the concept of point dissipativeness. Since the definition is meaningful in an arbitrary Banach space and also very useful, we give the precise definition and some of its extensions in such a space.

Definition 1. Let X be a Banach space. A continuous mapping $T : X \to X$ is said to be *point dissipative* if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $n_0 = n_0(x, B)$ such that $T^n x \in B$ for $n \geq n_0$. If moreover for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n(A) \subset B$ for $n \geq N(A)$, then T is said to be *compact dissipative*. If for any $x \in X$, there is an open neighborhood O_x and an integer $N(x)$ such that $T^n(O_x) \subset B$ for $n \geq N(x)$, then T is said to be *local dissipative*.

Obviously, local dissipative implies compact dissipative implies point dissipative.

For an ODE in $\mathbb{R}^N$ with ω-periodic coefficents, let $T : \mathbb{R}^N \to \mathbb{R}^N$ be the Poincaré map, that is, the mapping that takes an initial value $x \in \mathbb{R}^N$ to the value of the solution at time ω. Assuming that T is point dissipative, Levinson ([22]) proved that there is an integer $n^* \geq 1$ for which T^{n^*} has a fixed point, that is, there is an $n^*\omega$-periodic solution of the ODE. The proof uses the Brouwer fixed point theorem.

In his study of periodic solution of evolution equations, Browder ([2]) used fixed point theorems for nonexpansive maps to obtain results on the existence of ω-periodic solutions of a class of evolution equations with ω-periodic coefficients.

Concurrently, considerable effort was devoted to the discussion of periodic solutions for both periodically forced and autonomous delay differential equations with finite delay, and more generally to retarded functional differential equations with finite memory (RFDE).

Wright ([29]) studied the simplest type of nonlinear autonomous scalar delay differential equation

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)] \tag{1}$$

where $\alpha > 0$ is a constant. To obtain a function which satisfies this equation for $t = 0$, one must specify a function on the interval $[-1, 0]$. We choose a function φ from $C([-1, 0], \mathbb{R})$ and use (1) for $t \geq 0$ to define the function $x(t, \varphi)$, $t \geq 0$, which coincides with φ at $t = 0$.

By rescaling, the parameter α becomes the delay in which information about the past history influences the dynamics of the solutions of the equation. Wright ([29]) observed that, if α is small enough (namely, $\alpha \in (0, e^{-1})$), then every solution of (1) approaches the origin. However, as the delay is increased, the origin becomes unstable. In fact, for $\alpha = \pi/2$, there are two eigenvalues of the linearized equation on the imaginary axis. These eigenvalues are continuous in α and have real part > 0 for $\alpha > \pi/2$. This is a natural setting for a Hopf bifurcation, which at that time was unknown for such equations.

Jones ([18]) observed through numerical experiments that a nonconstant periodic solution of (1) occurred for all values of $\alpha > \pi/2$. He eventually proved that this was true. His method and ideas played an important role in the development of this subject. For a nonnegative increasing function $\varphi \in C([-1, 0], \mathbb{R})$ with $\varphi(-1) = 0$, Wright ([29]) proved that the solution $x(t, \varphi)$ of (1) has infinitely many zeroes $z_1(\varphi) < z_2(\varphi) < \cdots$ (such solutions are now referred to as slowly oscillating solutions). Moreover, in the interval $(z_2(\varphi), z_2(\varphi) + 1)$ the solution $x(t, \varphi)$ is increasing. Therefore, if we define $(T(\phi))(s) := x(z_2 + s, \phi)$ we see that the mapping T takes a positive cone into itself and that fixed points of T correspond to periodic solutions of period $z_2 + 1$. However, since the origin is on the boundary of the cone, one must devise a method for the elimination of the consideration of zero. To do this Jones ([18]) discovered an ejective fixed point theorem which made important use of the instability of the origin together with some a priori bounds on solutions.

The map T is the analogue of the Poincaré map for autonomous ODE which takes a 'section' in $\mathbb{R}^N$ to itself under the flow induced by the equation. The ejective fixed point theorem discovered by Jones was based on the following asymptotic fixed point theorem of Browder ([3]):

Theorem 1. *Let $S_0 \subset S_1 \subset S_2$ be convex sets of a Banach space X with S_1 and S_2 open and S_0 closed. If $T : S_2 \to X$ is a compact map such that, for some integer $m > 0$, T^m is well defined on S_1, $T^j(S_0) \subset S_1$, $0 \leq j \leq m$, and $T^m(S_1) \subset S_0$, then T has a fixed point.*

Motivated by a conversation and a letter from G. S. Jones, Browder ([4], [5]) proved further ejective fixed point theorems which are more appropriate to show existence of periodic solutions for autonomous equations.

In the case of periodically forced equations, the asymptotic fixed point theorem of Browder is very convenient because it does not require the existence of a convex invariant set (as the Schauder fixed point theorem does). For instance, in [28], Pliss discussed the behavior of the solutions of point dissipative ODE with periodic coefficients in the spirit of Levinson ([22]) and in one of the many results in his book, he makes use of the asymptotic fixed point theorem of Browder to prove

Theorem 2. *For a point dissipative ODE with coefficients ω-periodic in time there must be an ω-periodic solution; that is, $n^* = 1$ in Levinson's result.*

At this time, the theory of RFDE with finite delay was in the process of being developed. Many researchers began to discuss ω-periodic solutions of RFDE with finite delay. Using the asymptotic fixed point theorem of Browder for compact maps, many interesting results were given. However, it was necessary to assume that the Poincaré map was compact. This is only true in RFDE when the period is larger than the delay. This restriction appears for instance in the book by Yoshizawa ([30]) and it seemed unnatural, but no one could see what to do.

An important step (at least to us) was a theoretical investigation of Billotti and LaSalle ([1]) of the implication of point dissipativeness in the sense of Levinson on the dynamics of maps T for which some iterate T^{n_0} is compact. If T is the Poincaré map for a RFDE with ω-periodic coefficients, this will always occur. One of the important consequences of this theory was the following

Theorem 3. *If $T : X \to X$ is a continuous point dissipative map on a Banach space X for which there is an n_0 such that T^{n_0} is compact, then there exists the compact global attractor $\mathcal{A}$, that is, a compact, invariant set ($T\mathcal{A} = \mathcal{A}$) with the property that $\mathcal{A}$ attracts any bounded set $B \subset X$ in the sense that*

$$\lim_{n\to\infty} \operatorname{dist}_X(T^n B, \mathcal{A}) = \lim_{n\to\infty} \sup\{\operatorname{dist}_X(T^n(x), \mathcal{A}) : x \in B\} = 0.$$

This result, based only on point dissipativeness, used in a significant way the compactness of some iterate of T. However, it did not add any new information about the existence of ω-periodic solutions of RFDE for which the period of the coefficients is less than the delay.

In the meantime, Hale and Meyer ([15]) introduced a class of neutral functional differential equations (NFDE) for which the development of abstract theory led to further important concepts. To see how this class includes RFDE, we give a precise definition for a special case of NFDE with periodic coefficients since we are concentrating on fixed point theorems.

If $x : [-1, \infty) \to \mathbb{R}^N$ is a given function, define $x_t(\theta) = x(t + \theta)$ for each $\theta \in [-1, 0]$, $t \geq 0$. Let $C \equiv C([-1, 0], \mathbb{R}^N)$. Let $D : \varphi \in C \mapsto D\varphi \in \mathbb{R}^N$ be a continuous linear operator which is atomic at zero; that is, without loss of generality,

$$D\varphi = \varphi(0) - D_1\varphi, \tag{2}$$

where the function of bounded variation in the standard representation of the linear operator D_1 has no atom at zero. Also, suppose that $f : \mathbb{R} \times C \to \mathbb{R}^N$ is a continuous function which is ω-periodic in t. A NFDE is a differential relation

$$\frac{d}{dt} D(x_t) = f(t, x_t). \tag{3}$$

A solution of this equation with initial value $\varphi \in C$ at $t = 0$ is a continuous function $x : [-1, \alpha)$, $\alpha > 0$, which coincides with φ on $[-1, 0]$, has Dx_t differentiable for $t \geq 0$ (the right hand derivative at $t = 0$) and satisfies (3). It is not assumed that x is diffferentiable, but that Dx_t is. This is like a weak solution. In fact, some of these equations occur in the applications through the wave equation on a bounded interval with dynamic boundary conditions which are periodic in time. For a discussion of such equations and their importance, see, for example, Hale and Lunel ([14]).

Assuming that solutions exist for $t \geq 0$, we can define the Poincaré map T on C that takes the initial value $\varphi \in C$ to the value of the solution $x_t(\cdot, \varphi)$ at $t = \omega$.

If $D_1 = 0$ in (2), we obtain RFDE and some iterate of the map T is compact. However, in general, no iterate of the map T with $D_1 \neq 0$ ever becomes compact.

To obtain the beginnings of a qualitative theory for (3), Hale and Meyer ([15]) made further restrictions on D; namely, they supposed that the operator D *is exponentially stable*, that is, if $C_0 = \{\varphi \in C : D\varphi = 0\}$, then each solution of the functional equation

$$Dy_t = 0 \tag{4}$$

with initial value $\varphi \in C_0$ approaches zero exponentially as $t \to \infty$.

Later, it was shown by Cruz and Hale ([8]) and Henry ([16]) that *the period map T for* (3) *is an α-contraction if D is exponentially stable*. The term α-contraction is in the sense of Kuratowski (see [13]) for instance) and it will be defined later in this paper. For RFDE, this result implies that the Poincaré map T is an α-contraction for any $\omega > 0$. Once we state a general fixed point theorem for such maps T, we have shown that the restriction that the period is larger than the delay is unnecessary. Notice that, in general, even in the case of exponentially stable D operators, no iterate of T is compact (in fact, T can be a homeomorphism).

Motivated by these equations as well as certain hyperbolic PDE, Hale, LaSalle and Slemrod ([12]) investigated the dynamical implications of maps T which are α-contractions on a Banach space X and satisfy some type of dissipative property. They also investigated conditions for the existence of maximal compact invariant sets for maps T which are only continuous. To state the results, it is convenient to introduce some additional terminology.

Definition 2. Suppose that T is a continuous map on a Banach space X. If $\mathcal{S}$ is a collection of subsets of X, then a subset J of X is said to *attract* $\mathcal{S}$ *under* T (or, more briefly, *attract* $\mathcal{S}$) if, for each $K \in \mathcal{S}$,

$$\lim_{n\to\infty} \operatorname{dist}_X(T^n K, J) = 0.$$

Definition 3. A continuous map T on a Banach space X is *asymptotically smooth* if, for any bounded set $B \subset X$ for which the positive orbit $\gamma^+(B)$ under the action of T is bounded, there is a compact set $J \subset X$ which attracts B; in particular, the ω-limit set of B is a compact set.

Definition 4. If T is a continuous map on a Banach space X and $A \subset X$ is a bounded invariant set, then A is *Lyapunov stable* if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\operatorname{dist}(x, A) < \delta$ implies $\operatorname{dist}(T^n(x), A) < \epsilon$ for any $n \geq 0$; A is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that $\operatorname{dist}(x, A) < \eta$ implies $\lim_{n\to\infty} \operatorname{dist}(T^n(x), A) = 0$; and A is a *local attractor* or is *uniformly asymptotically stable* if there is an open neighborhood U of A such A attracts U.

One of the important results in Hale, LaSalle and Slemrod ([12]) is the following

Theorem 4. *Suppose that $T : X \to X$ is continuous and there is a compact set K which attracts compact sets of X. Then $A = \bigcap_{n\geq 0} T^n K$ is independent of K and satisfies the following*

(i) *A is the maximal compact invariant set.*
(ii) *A is Lyapunov stable.*
(iii) *A attracts compact sets of X.*
(iv) *For any compact set J in X, there is a neighorhood $U(J)$ of J such that $\gamma^+(U(J))$ is bounded.*
(v) *If, in addition, T is asymptotically smooth, then, for any compact set $K \subset X$, there is a neighborhood $U(K)$ such that $\mathcal{A}$ attracts $U(K)$; in particular, $\mathcal{A}$ is a local attractor.*
(vi) *If T is asymptotically smooth and, in addition, the positive orbit of any bounded set $B \subset X$ is bounded, then the compact global attractor exists.*

Remark 1. We will use later an observation of Cooperman ([7]) that, *if T is asympotically smooth and compact dissipative, then there is a compact invariant set of T which attracts compact sets of X.*

A very important class of maps we will be considering is the set of α-*contractions*. To define α-contraction, we first introduce the Kuratowski measure of noncompactness ([21]).

Definition 5. If $A \subset X$ is a bounded set, then the *Kuratowski measure of noncompactness* $\alpha(A)$ of A is the infimum of the numbers d such that A can be covered by a finite number of sets with diameter less than d.

With this definition, Kuratowski proved that, if F_n is a decreasing sequence of closed bounded sets of a Banach space X and $\alpha(F_n)$ tends to zero, then the set $\bigcap_{n\geq 1} F_n$ is nonempty and compact. This is a unification of classical results about the intersection of a decreasing sequence of compact sets and of a decreasing sequence of closed bounded sets whose diameter tends to zero.

Definition 6. A map $T : X \to X$ is an *α-contraction* if there a number k, with $0 \leq k < 1$, such that for any bounded set $A \subset X$, $T(A)$ is also bounded and $\alpha(T(A)) \leq k\alpha(A)$; T is *α-condensing* if for any bounded set A with $\alpha(A) > 0$, $T(A)$ is also bounded and $\alpha(T(A)) < \alpha(A)$.

It is known that α-contractions and α-condensing maps are asymptotically smooth (see [11]). Notice that it is assumed that the fundamental hypothesis in Theorem 4 is different from the one of Billotti and LaSalle ([1]). In fact, they assumed only that the map was point dissipative and some iterate of T was compact. It is not difficult to show that this hypothesis implies the hypotheses imposed in Theorem 4. Theorem 4 is a significant generalization of that of Billotti and LaSalle ([1]).

From the above discussion, it is clear that an asymptotic fixed point theorem for a general class which includes α-contractions would be very useful. The extent to which one can obtain such results using only point dissipativeness is an open problem. To obtain such results, one may have to consider special additional properties of the maps T that arise through the discussion of particular types of evolutionary equations. We say more about this later.

In the case of retarded functional equations the first result about the existence of a forced periodic solution that does not assume that the period is larger than the delay is due to Jones ([19]). He proves a theorem similar to Theorem 1 but he replaces the compactness of the map by the compactness of some sets. His result deals specifically with the space $C([-r, 0], \mathbb{R}^N)$. A more abstract theorem is a result of Horn ([17]) which we state as

Theorem 5 (Horn [17]). *Suppose that $S_0 \subset S_1 \subset S_2$ are convex sets of a Banach space X with S_0, S_2 compact and S_1 open in S_2. Let $T : S_2 \to X$ be a continuous map such that, for some integer $m > 0$, $T^j(S_1) \subset S_2$ for $0 \leq j \leq m-1$, and $T^j(S_1) \subset S_0$ for $m \leq j \leq 2m-1$. Then T has a fixed point.*

Using Horn's fixed point theorem, we can prove the existence of a periodic solution for dissipative retarded equations without assuming that the period is larger than the delay. We simply take the space X as $C([-r, 0], \mathbb{R}^N)$ and S_2 is the intersection of a certain ball in X with the set of functions which have a certain Lipschitz constant K. In the case of neutral equations it seems to be difficult to exhibit such S_2. For α-contractions the fixed point theorem of Darbo ([10]) was already known. Therefore, it was very natural to look for an asymptotic fixed point theorem for α-contractions. In that direction, the following result was obtained independently by Hale and Lopes ([13]) and Nussbaum ([25]–[27]) using different methods.

Theorem 6. *If X is a Banach space, and $T : X \to X$ is a continuous, α-condensing map which is compact dissipative, then there is a fixed point of T.*

Corollary 1. *If X is a Banach space, and $T : X \to X$ is a continuous, point dissipative α-condensing map for which the positive orbit of any compact set is bounded, then T has a fixed point.*

Proof. As we have pointed out, if T is α-condensing then it is asymptotically smooth. Moreover, point dissipative asymptotically smooth maps for which the positive orbit of each compact set is bounded are compact dissipative (this follows from the discussion on pages 18 and 19 of [11]), and this proves the corollary.

Corollary 2. *If X is a Banach space, $T : X \to X$ is a continuous, α-condensing map which is point dissipative, and there is a compact invariant set J which is Lyapunov stable, then T has a fixed point.*

Proof. If T is point dissipative and $J \subset X$ is invariant and stable, then J attracts compact sets of X, and T is compact dissipative. The result follows from Theorem 6.

The proof of Hale and Lopes ([13]) made extensive use of the detailed dynamical results of Hale, LaSalle and Slemrod ([12]), an interesting dynamical lemma and Horn's theorem. We give some of the details of this method since the lemma has led to other classes of maps for which there are asymptotic fixed point theorems. On the other hand, in [25]–[27], Nussbaum develops a degree theory for α-contractions and α-condensing maps, while in our approach the degree theory is hidden in Horn's Theorem 5. Although Theorem 6 is enough for the applications we have in mind, we could not recover some results proved by the degree theory. We will come back to this point later in this paper.

We state first an implication of compact dissipativeness for asymptotically smooth maps.

Lemma 1. *If T is a continuous map on a Banach space X which is asymptotically smooth and compact dissipative, then there exist convex sets $K \subset B \subset S$ in X with K compact and attracting compact sets of X, S closed and bounded, and B open in S.*

Proof. From the hypothesis, there is a compact invariant set J of X such that J attracts compact sets of X. Let $\mathcal{A}$ be the maximal compact set in Theorem 4 and let $K = \overline{\text{co}}\,\mathcal{A}$. Then K attracts compact sets of X. Theorem 4(v) implies that there is an open convex neighborhood B of K such that $\gamma^+(B)$ is bounded and K attracts B. If $S = \overline{\text{co}}\,\gamma^+(B)$, then the conclusion of the lemma is proved.

An interesting lemma of Hale and Lopes ([13]) whose proof can be found in [13] is the following.

Lemma 2. *Suppose that $K \subset B \subset S$ are convex sets of a Banach space X with K compact, S closed and bounded, and B open in S. If $T : S \to X$ is continuous, $\gamma^+(B) \subset S$, and K attracts each point of B, then there is a closed bounded subset J of S such that*

$$J = \overline{\mathrm{co}}\, \gamma^+(T(B \cap J)), \qquad J \cap K \neq \emptyset, \tag{5}$$

where co *denotes convex hull and γ^+ the positive orbit. If, in addition, K attracts compact subsets of B, and J is compact, then T has a fixed point.*

According to Lemma 2, if a certain map T satisfies some assumptions and we wish to prove the existence of a fixed point, we have to show that any set J satisfying the set equation (5) is compact. Although this is far from being always easy, we give some examples for which that property can be verified. Under the assumptions of Theorem 6, this can be done easily. Later we give some examples for which we have been unable to do it.

Proof of Theorem 6. If K, B, S are chosen as in Lemma 2, then K attracts compact sets of X. A rather elementary argument shows that the set J in (5) is compact and the theorem is proved.

P. Magal and X. Zhao ([23]) discovered a fixed point theorem making use of Lemma 2 which could be used effectively in the discussion of population models (see also X. Zhao [31]). To describe their result, we need the following definition.

Definition 7. Let X be a Banach space and $T : X \to X$ a continuous map. If, for each $B \subset X$, we define $\hat{T}(B) = \overline{\mathrm{co}}(T(B))$, then we say that T is *convex α-contracting* if $\lim_{n\to\infty} \alpha(\hat{T}^n(B)) = 0$ for every bounded $B \subset X$.

Theorem 7 (Asymptotic fixed point theorem for convex α-contracting maps). *If T is convex α-contracting on X and there is a compact invariant set which is stable and attracts points of X, then T has a fixed point.*

Proof. From Lemma 1, we can construct sets $K \subset B \subset S$ so that the conditions of Lemma 2 are satisfied, and thus there is a set J such that

$$J = \overline{\mathrm{co}}\, \tilde{J}, \qquad \tilde{J} = \gamma^+(T(B \cap J)).$$

Since $\tilde{J} = T(B \cap J) \cup T(\tilde{J})$, and $J = \overline{\mathrm{co}}\, \tilde{J}$, it follows that $\tilde{J} \subset T(J)$. Thus,

$$J \subset \hat{T}(J) \subset \cdots \subset \hat{T}^n(J), \quad n \geq 0.$$

Since T is convex α-contracting, it follows that $0 \leq \alpha(J) \leq \alpha(\hat{T}^n(J)) \to 0$ as $n \to \infty$. Therefore, $\alpha(J) = 0$ and J is compact. Lemma 2 implies the existence of a fixed point of T and the theorem is proved.

Under more restrictive assumptions, Magal and Zhao ([23]) proved Theorem 7 for point dissipative maps. Magal and Zhao were led to discuss maps of this type in their study of persistence in age dependent population models.

The construction of Definition 5 had been used already in [26] and it was indicated why it is important to consider the more general class. With the degree theory developed in [27], one obtains a fixed point theorem which is more general than Theorem 7.

It is also not known if one can deduce that there is a fixed point of an asymptotically smooth map which is compact dissipative.

A more general class for which Lemma 2 applies has been introduced by S. J. Daher ([9]) (see also I.-S. Kim [20]). To introduce it, we first need

Definition 8. If $T : X \to X$ is a continuous map in a Banach space X, for any compact subset K of X we let

$$K_0 = \mathrm{co}(K), \qquad K_n = \mathrm{co}(T(K_{n-1})), \qquad K_T = \bigcup_{n \geq 0} K_n.$$

We say that T is *sequentially α-condensing* if for any compact set K for which $\alpha(K_T) \leq \alpha(T(K_T))$ the set $T(K_T)$ is relatively compact.

In [9] and [20] it is proved that if T is sequentially α-condensing then the set J satisfying (5) is compact. Therefore, for such maps the existence of fixed points follows.

Cholewa and Hale ([6]) have given results on the classification of those maps for which compact dissipative and point dissipative are equivalent. It remains to prove that there will be a fixed point of these maps under the assumption of compact dissipative.

Massatt [24] has proved the following interesting result:

Theorem 8. *For NFDE with an exponentially stable D operator, point dissipative and compact dissipative are equivalent. Therefore, the Poincaré map T has a fixed point if it is point dissipative.*

The method of proof should be applicable to other types of equations that occur frequently in applications. Massatt's proof involves the application of a theory of operators which are dissipative in two spaces, one compactly embedded in the other. For NFDE, the two spaces are $C([-1,0],\mathbb{R}^N)$ and $W^{1,\infty}([0,1],\mathbb{R}^N)$. For this case, the Poincaré mapping is an α-contraction in both spaces.

In general, it is much easier to verify point dissipativeness in a particular example.

It would be interesting to pursue this topic in more detail for particular types of evolutionary equations.

The fixed points theorems obtained by our approach have been enough to cover all applications we have encountered so far (see [11] and some references therein). However, there are interesting fixed point theorems proved in [25]–[27] that have been out of our reach. For instance, concerning Theorem 6, Nussbaum assumes that the map T is an α-contraction in an open neighborhood of the attractor only. Moreover, since his proof of the fixed point property follows from a degree theory, it is preserved by a small perturbation in some class of maps.

Another interesting result of Nussbaum is the following: Let B a closed ball in a real Banach space X and $f : B \to B$ a continuous map. Assume that there exist a constant $k < 1$ and a compact set $K \subset X$ such that $d(f(x), K) \leq kd(x, k)$ for all $x \in B$, where $d(y, K)$ denotes the distance from a point y to K. Then f has a fixed point.

In such a case the set equation (5) becomes $J = \overline{\text{co}}(T(J))$. Therefore, it is natural to ask if under the assumptions of the theorem such a J is compact.

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